

# On the Breiman conjecture

Péter Kevei\*

David M. Mason†

August 10, 2015

## Abstract

Let  $Y_1, Y_2, \dots$  be positive, nondegenerate, i.i.d.  $G$  random variables, and independently let  $X_1, X_2, \dots$  be i.i.d.  $F$  random variables. In this note we show that whenever  $\sum X_i Y_i / \sum Y_i$  converges in distribution to nondegenerate limit for some  $F \in \mathcal{F}$ , in a specified class of distributions  $\mathcal{F}$ , then  $G$  necessarily belongs to the domain of attraction of a stable law with index less than 1. The class  $\mathcal{F}$  contains those nondegenerate  $X$  with a finite second moment and those  $X$  in the domain of attraction of a stable law with index  $1 < \alpha < 2$ .

## 1 Introduction and results

Let  $Y, Y_1, \dots$  be positive, nondegenerate, i.i.d. random variables with distribution function [df]  $G$ , and independently let  $X, X_1, \dots$  be i.i.d. nondegenerate random variables with df  $F$ . Let  $\phi_X$  denote the characteristic function [cf] of  $X$ . We shall use the notation  $Y \in D(\beta)$  to mean that  $Y$  is in the domain of attraction of a stable law of index  $0 < \beta < 1$ , and  $Y \in D(0)$  will denote that  $1 - G$  is slowly varying at infinity. Furthermore  $\mathcal{RV}_\infty(\rho)$  will signify the class of positive measurable functions regularly varying at infinity with index  $\rho$ , and  $\mathcal{RV}_0(\rho)$  the class of positive measurable functions regularly varying at zero with index  $\rho$ . In particular, using this notation  $Y \in D(\beta)$ , with  $0 \leq \beta < 1$ , if and only if  $1 - G \in \mathcal{RV}_\infty(-\beta)$ .

For each integer  $n \geq 1$  set

$$T_n = \sum_{i=1}^n X_i Y_i / \sum_{i=1}^n Y_i. \quad (1)$$

Notice that  $\mathbb{E}|X| < \infty$  implies that  $T_n$  is stochastically bounded. Theorem 4 of Breiman [2] says that  $T_n$  converges in distribution along the full sequence  $\{n\}$  for *every*  $X$  with finite expectation, and with at least one limit law being nondegenerate if and only if

$$Y \in D(\beta), \text{ with } 0 \leq \beta < 1. \quad (2)$$

---

\*Center for Mathematical Sciences, Technische Universität München, Boltzmannstraße 3, 85748 Garching, Germany, [peter.kevei@tum.de](mailto:peter.kevei@tum.de)

†Department of Applied Economics and Statistics, University of Delaware, 213 Townsend Hall, Newark, DE 19716, USA, [davidm@udel.edu](mailto:davidm@udel.edu)

Let  $\mathcal{X}$  denote the class of nondegenerate random variables  $X$  with  $\mathbb{E}|X| < \infty$  and let  $\mathcal{X}_0$  denote those  $X \in \mathcal{X}$  such that  $\mathbb{E}X = 0$ . At the end of his paper Breiman conjectured that if for *some*  $X \in \mathcal{X}$ ,  $T_n$  converges in distribution to some nondegenerate random variable  $T$ , written

$$T_n \rightarrow_d T, \text{ as } n \rightarrow \infty, \text{ with } T \text{ nondegenerate,} \quad (3)$$

then (2) holds. By Proposition 2 and Theorem 3 of [2], for any  $X \in \mathcal{X}$ , (2) implies (3), in which case  $T$  has a distribution related to the arcsine law. Using this fact, we see that his conjecture can restated to be: for any  $X \in \mathcal{X}$ , (2) is equivalent to (3).

It has proved to be surprisingly challenging to resolve. Mason and Zinn [8] partially verified Breiman's conjecture. They established that whenever  $X$  is nondegenerate and satisfies  $\mathbb{E}|X|^p < \infty$  for some  $p > 2$ , then (2) is equivalent to (3). In this note we further extend this result.

**Theorem** *Assume that for some  $X \in \mathcal{X}_0$ ,  $1 < \alpha \leq 2$ , positive slowly varying function  $L$  at zero and  $c > 0$ ,*

$$\frac{-\log(\Re \phi_X(t))}{|t|^\alpha L(|t|)} \rightarrow c, \text{ as } t \rightarrow 0, \quad (4)$$

*(in the case  $\alpha = 2$  we assume that  $\liminf_{t \searrow 0} L(t) > 0$ ). Whenever (3) holds then  $Y \in D(\beta)$  for some  $\beta \in [0, 1)$ .*

Let  $\mathcal{F}$  denote the class of random variables that satisfy the conditions of the theorem. Applying our theorem in combination with Proposition 2 and Theorem 3 of [2] we get the following corollary.

**Corollary** *Whenever  $X - \mathbb{E}X \in \mathcal{F}$ , (2) is equivalent to (3).*

**Remark 1** It can be inferred from Theorem 8.1.10 of Bingham et al. [1] that for  $X \in \mathcal{X}_0$ , (4) holds for some  $1 < \alpha < 2$ , positive slowly varying function  $L$  at zero and  $c > 0$  if and only if  $X$  satisfies  $\mathbb{P}\{|X| > x\} \sim L(1/x)x^{-\alpha}c\Gamma(\alpha)\frac{2}{\pi}\sin(\frac{\pi\alpha}{2})$ . Note that a random variable  $X \in \mathcal{X}_0$  in the domain of attraction of a stable law of index  $1 < \alpha < 2$  satisfies (4). Also a random variable  $X \in \mathcal{X}_0$  with variance  $0 < \sigma^2 < \infty$  fulfills (4) with  $\alpha = 2$ ,  $L = 1$  and  $c = \sigma^2/2$ .

**Remark 2** Consult Kevei and Mason [7] for a fairly exhaustive study of the asymptotic distributions of  $T_n$  along subsequences, along with revelations of their unexpected properties.

The theorem follows from the two propositions below. First we need more notation. For any  $\alpha \in (1, 2]$  define for  $n \geq 1$

$$S_n(\alpha) = \frac{\sum_{i=1}^n Y_i^\alpha}{(\sum_{i=1}^n Y_i)^\alpha}. \quad (5)$$

**Proposition 1** *Assume that the assumptions of the theorem hold. Then for some  $0 < \gamma \leq 1$*

$$\mathbb{E}S_n(\alpha) \rightarrow \gamma, \text{ as } n \rightarrow \infty. \quad (6)$$

The next proposition is interesting in its own right. It is an extension of Theorem 5.3 by Fuchs et al. [4], where  $\alpha = 2$  (see also Proposition 3 of [8]).

**Proposition 2** *If (6) holds with some  $\gamma \in (0, 1]$  then  $Y \in D(\beta)$ , for some  $\beta \in [0, 1)$ , where  $-\beta \in (-1, 0]$  is the unique solution of*

$$\text{Beta}(\alpha - 1, \beta + 1) = \frac{\Gamma(\alpha - 1)\Gamma(1 + \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.$$

*In particular,  $Y \in D(0)$  for  $\gamma = 1$ .*

*Conversely, if  $G \in D(\beta)$ ,  $0 \leq \beta < 1$ , then (6) holds with*

$$\gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 + \beta)} = \frac{1}{(\alpha - 1)\text{Beta}(\alpha - 1, \beta + 1)}.$$

## 2 Proofs

Set for each  $n \geq 1$ ,  $R_i = Y_i / \sum_{l=1}^n Y_l$ , for  $i = 1, \dots, n$ . For notational ease we drop the dependence of  $R_i$  on  $n \geq 1$ . Consider the sequence of strictly decreasing continuous functions  $\{\varphi_n\}_{n \geq 1}$  on  $[1, \infty)$  defined by  $\varphi_n(y) = \mathbb{E}(\sum_{i=1}^n R_i^y)$ ,  $y \in [1, \infty)$ . Note that each function  $\varphi_n$  satisfies  $\varphi_n(1) = 1$ . By a diagonal selection procedure for each subsequence of  $\{n\}_{n \geq 1}$  there is a further subsequence  $\{n_k\}_{k \geq 1}$  and a right continuous nonincreasing function  $\psi$  such that  $\varphi_{n_k}$  converges to  $\psi$  at each continuity point of  $\psi$ .

**Lemma 1** *Each such function  $\psi$  is continuous on  $(1, \infty)$ .*

*Proof* Choose any subsequence  $\{n_k\}_{k \geq 1}$  and a right continuous nonincreasing function  $\psi$  such that  $\varphi_{n_k}$  converges to  $\psi$  at each continuity point of  $\psi$  in  $(1, \infty)$ . Select any  $x > 1$  and continuity points  $x_1, x_2 \in (1, \infty)$  of  $\psi$  such that  $1 < x_1 < x < x_2 < \infty$ . Set  $\rho_1 = x_1 - 1$  and  $\rho_2 = x_2 - 1$ . Since  $\rho_2/\rho_1 > 1$  we get by Hölder's inequality

$$\sum_{i=1}^{n_k} R_i^{x_1} = \sum_{i=1}^{n_k} R_i^{\rho_1} R_i \leq \left( \sum_{i=1}^{n_k} R_i^{\rho_2} R_i \right)^{\rho_1/\rho_2} = \left( \sum_{i=1}^{n_k} R_i^{x_2} \right)^{\rho_1/\rho_2}.$$

Thus by taking expectations and using Jensen's inequality we get  $\varphi_{n_k}(x_1) \leq (\varphi_{n_k}(x_2))^{\rho_1/\rho_2}$ . Letting  $n_k \rightarrow \infty$ , we have  $\psi(x_1) \leq (\psi(x_2))^{\rho_1/\rho_2}$ . Since  $x_1 < x$  and  $x_2 > x$  can be chosen arbitrarily close to  $x$  we conclude by right continuity of  $\psi$  at  $x$  that  $\psi(x-) = \psi(x+) = \psi(x)$ .  $\square$

*Proof of Proposition 1* For a complex  $z$ , we use the notation for the principal branch of the logarithm,  $\text{Log}(z) = \log|z| + \imath \arg z$ , where  $-\pi < \arg z \leq \pi$ , i.e.  $z = |z| \exp(\imath \arg z)$ . We see that for all  $t$

$$\begin{aligned} \mathbb{E} \exp(\imath t T_n) &= \mathbb{E} \left( \prod_{j=1}^n \phi_X(t R_j) \right) \\ &= \mathbb{E} \left( \prod_{j=1}^n \exp(\text{Log} \phi_X(t R_j)) \right). \end{aligned}$$

Since  $\mathbb{E}X = 0$  we have  $\Re \phi_X(u) = 1 - o_+(u)$ , where  $o_+(u) \geq 0$ , and  $o_+(u)$  and  $o_+(u)/u \rightarrow 0$  as  $u \rightarrow 0$ ; and  $\Im \phi_X(u) = o(u)$ . This when combined with

$$(\arctan \theta)' = \frac{1}{1 + \theta^2}$$

gives as  $u \rightarrow 0$ ,

$$\arg \phi_X(u) = \arctan \left( \frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right) = o(u).$$

Note that for all  $|u| > 0$  sufficiently small so that  $\Re \phi_X(u) > 0$

$$\text{Log} \phi_X(u) = \text{Log}(\Re \phi_X(u) + i \Im \phi_X(u)) = \log \Re \phi_X(u) + \text{Log} \left( 1 + i \frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right),$$

where for the second term

$$\Re \text{Log} \left( 1 + i \frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right) = \frac{1}{2} \left( \frac{\Im \phi_X(u)}{\Re \phi_X(u)} \right)^2 (1 + o(u)), \text{ as } u \rightarrow 0.$$

Thus for every  $\varepsilon > 0$  for all  $|t| > 0$  sufficiently small and independent of  $n \geq 1$  and  $R_1, \dots, R_n$

$$1 - \varepsilon^2 t^2 \leq \cos(\varepsilon t) \leq \Re \left( \exp \left\{ \sum_{j=1}^n \text{Log} \left( 1 + i \frac{\Im \phi_X(tR_j)}{\Re \phi_X(tR_j)} \right) \right\} \right) \leq e^{2^{-1} \varepsilon t^2} \leq 1 + \varepsilon t^2.$$

Thus we obtain

$$\begin{aligned} \mathbb{E} \exp \left\{ \sum_{j=1}^n \log \Re \phi_X(tR_j) \right\} (1 - \varepsilon^2 t^2) &\leq \mathbb{E} (\Re \exp(itT_n)) \\ &= \Re \mathbb{E} \exp(itT_n) \\ &\leq \mathbb{E} \exp \left\{ \sum_{j=1}^n \log \Re \phi_X(tR_j) \right\} (1 + \varepsilon t^2). \end{aligned}$$

We shall show (4) implies that (6) holds for some  $0 < \gamma \leq 1$ . Now using (4) we get for any  $0 < \delta < c$  and all  $|t|$  small enough independent of  $n \geq 1$ ,

$$\begin{aligned} -\varepsilon t^2 + \log \mathbb{E} \exp \left( -(c + \delta) |t|^\alpha \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) &\leq \log (\Re \mathbb{E} \exp(itT_n)) \\ &\leq \varepsilon t^2 + \log \mathbb{E} \exp \left( -(c - \delta) |t|^\alpha \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right). \end{aligned}$$

Next since  $\log s/(1 - s) \rightarrow -1$  as  $s \nearrow 1$ , for all  $|t|$  small enough independent of  $n \geq 1$  and  $R_1, \dots, R_n$ , (keeping mind that  $\sum_{i=1}^n R_i = 1$  and  $1 < \alpha \leq 2$ )

$$\begin{aligned} &\log \mathbb{E} \exp \left( -(c + \delta) |t|^\alpha \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \\ &\geq - \left( 1 + \frac{\delta}{2} \right) \mathbb{E} \left( 1 - \exp \left( -(c + \delta) |t|^\alpha \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \right) \end{aligned}$$

and

$$\begin{aligned} & \log \mathbb{E} \exp \left( - (c - \delta) |t|^\alpha \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \\ & \leq - \left( 1 - \frac{\delta}{2} \right) \mathbb{E} \left( 1 - \exp \left( - (c - \delta) |t|^\alpha \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \right). \end{aligned}$$

Further since  $(1 - \exp(-y))/y \rightarrow 1$  as  $y \searrow 0$ , for all  $|t|$  small enough independent of  $n \geq 1$ ,

$$\begin{aligned} & - \left( 1 + \frac{\delta}{2} \right) \mathbb{E} \left( 1 - \exp \left( - (c + \delta) |t|^\alpha \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \right) \\ & \geq - (1 + \delta) (c + \delta) |t|^\alpha \mathbb{E} \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \end{aligned}$$

and

$$\begin{aligned} & - \left( 1 - \frac{\delta}{2} \right) \mathbb{E} \left( 1 - \exp \left( - (c - \delta) |t|^\alpha \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \right) \right) \\ & \leq - (1 - \delta) (c - \delta) |t|^\alpha \mathbb{E} \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right). \end{aligned}$$

Therefore for all  $|t|$  small enough independent of  $n$ ,

$$\begin{aligned} & - \varepsilon t^2 - (1 + \delta) (c + \delta) |t|^\alpha \mathbb{E} \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right) \\ & \leq \log (\Re \mathbb{E} \exp (itT_n)) \\ & \leq \varepsilon t^2 - (1 - \delta) (c - \delta) |t|^\alpha \mathbb{E} \left( \sum_{i=1}^n R_i^\alpha L(|t| R_i) \right). \end{aligned}$$

By the Potter's bound, Theorem 1.5.6 (i) in [1], for all  $A > 1$  and  $1 < \alpha_1 < \alpha < \alpha_2$ , for all  $t > 0$  small enough independent of  $n \geq 1$ ,

$$A^{-1} \sum_{i=1}^n R_i^{\alpha_2} \leq \sum_{i=1}^n R_i^\alpha L(|t| R_i) / L(|t|) \leq A \sum_{i=1}^n R_i^{\alpha_1}. \quad (7)$$

We see now that for all  $n \geq 1$  and  $0 < 4\varepsilon < c$ , appropriate  $1 < \alpha_1 < \alpha < \alpha_2$  and all  $|t|$  small

enough independent of  $n$ ,

$$\begin{aligned}
& -\varepsilon t^2 - (1 + \varepsilon)(c + 2\varepsilon)|t|^\alpha L(|t|) \mathbb{E} S_n(\alpha_2) \\
& = -\varepsilon t^2 - (1 + \varepsilon)(c + 2\varepsilon)|t|^\alpha L(|t|) \mathbb{E} \left( \sum_{i=1}^n R_i^{\alpha_2} \right) \\
& \leq \log(\Re \mathbb{E} \exp(itT_n)) \\
& \leq \varepsilon t^2 - (1 - \varepsilon)(c - 2\varepsilon)|t|^\alpha L(|t|) \mathbb{E} \left( \sum_{i=1}^n R_i^{\alpha_1} \right) \\
& = \varepsilon t^2 - (1 - \varepsilon)(c - 2\varepsilon)|t|^\alpha L(|t|) \mathbb{E} S_n(\alpha_1).
\end{aligned}$$

Choose any subsequence  $\{n_k\}_{k \geq 1}$  and a right continuous nonincreasing function  $\psi$  such that  $\varphi_{n_k}$  converges to  $\psi$  at each continuity point of  $\psi$ , which by Lemma 1 above is all  $(1, \infty)$ . We see that  $\mathbb{E} S_{n_k}(\alpha) \rightarrow \psi(\alpha)$ ,  $\mathbb{E} S_{n_k}(\alpha_1) \rightarrow \psi(\alpha_1)$  and  $\mathbb{E} S_{n_k}(\alpha_2) \rightarrow \psi(\alpha_2)$ , where necessarily  $0 < \psi(\alpha_2) \leq \psi(\alpha) \leq \psi(\alpha_1) \leq 1$ . (The case  $\psi(\alpha_1) = 0$  cannot happen, since this would imply that  $T$  is degenerate.) We see that for all  $|t|$  sufficiently small independent of  $n_k \geq 1$ ,

$$-\varepsilon - (1 + \varepsilon)(c + 3\varepsilon)\psi(\alpha_2) \leq \log(\Re \mathbb{E} \exp(itT_{n_k})) / (|t|^\alpha L(|t|)) \leq \varepsilon - (1 - \varepsilon)(c - 3\varepsilon)\psi(\alpha_1),$$

where for  $\alpha = 2$  we use the assumption that in this case  $\liminf_{t \searrow 0} L(t) > 0$ . Since  $0 < 4\varepsilon < c$  can be made arbitrarily small and  $0 \leq \psi(\alpha_1) - \psi(\alpha_2)$  can be made as close to zero as desired, by letting  $n_k \rightarrow \infty$ , we get that for all  $|t|$  sufficiently small

$$-\varepsilon - (1 + \varepsilon)(c + 4\varepsilon)\psi(\alpha) \leq \log(\Re \mathbb{E} \exp(itT)) / (|t|^\alpha L(|t|)) \leq \varepsilon - (1 - \varepsilon)(c - 4\varepsilon)\psi(\alpha),$$

which can happen only if  $\psi(\alpha)$  does not depend on  $\{n_k\}$ . Thus (6) holds for some  $0 < \gamma \leq 1$ , namely  $\gamma = \psi(\alpha)$ .  $\square$

*Proof of Proposition 2* To begin with, we note that whenever (6) holds, necessarily  $\mathbb{E} Y = \infty$ . To see this, write  $D_n^{(1)} = \max_{1 \leq i \leq n} Y_i / (\sum_{i=1}^n Y_i)$  and observe that

$$\begin{aligned}
\left(D_n^{(1)}\right)^\alpha &= \max_{1 \leq i \leq n} \frac{Y_i^\alpha}{(\sum_{i=1}^n Y_i)^\alpha} \leq S_n(\alpha) \\
&\leq \max_{1 \leq i \leq n} \frac{Y_i^{\alpha-1}}{(\sum_{i=1}^n Y_i)^{\alpha-1}} = \left(D_n^{(1)}\right)^{\alpha-1}.
\end{aligned}$$

From these inequalities it is easy to prove that  $\mathbb{E} S_n(\alpha) \rightarrow 0$ ,  $n \rightarrow \infty$ , if and only if

$$D_n^{(1)} \rightarrow_P 0, \quad n \rightarrow \infty. \tag{8}$$

Proposition 1 of Breiman [2] says that (8) holds if and only there exists a sequence of positive constants  $B_n$  converging to infinity such that

$$\sum_{i=1}^n Y_i / B_n \rightarrow_P 1, \quad n \rightarrow \infty. \tag{9}$$

Since  $\mathbb{E}Y < \infty$  obviously implies (9), it readily follows that  $\mathbb{E}S_n(\alpha) \rightarrow 0$ ,  $n \rightarrow \infty$ , and thus (6) cannot hold.

We shall first prove the first part of Proposition 2. Following similar steps as in [8] we have that

$$\begin{aligned}\mathbb{E} \frac{\sum_{i=1}^n Y_i^\alpha}{(\sum_{i=1}^n Y_i)^\alpha} &= n \mathbb{E} \frac{Y_1^\alpha}{(\sum_{i=1}^n Y_i)^\alpha} \\ &= \frac{n}{\Gamma(\alpha)} \mathbb{E} \int_0^\infty Y_1^\alpha e^{-t \sum_{i=1}^n Y_i} t^{\alpha-1} dt \\ &= \frac{n}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathbb{E} (e^{-tY_1} Y_1^\alpha) (\mathbb{E} e^{-tY_1})^{n-1} dt \\ &=: \frac{n}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \phi_\alpha(t) \phi_0(t)^{n-1} dt.\end{aligned}$$

Next, assuming (6) and arguing as in the proof of Theorem 3 in [2] we get

$$s \int_0^\infty t^{\alpha-1} \phi_\alpha(t) e^{s \log \phi_0(t)} dt \rightarrow \gamma \Gamma(\alpha), \quad s \rightarrow \infty. \quad (10)$$

For  $y \geq 0$ , let  $q(y)$  denote the inverse of  $-\log \phi_0(t)$ . Changing the variables to  $y = -\log \phi_0(t)$  and  $t = q(y)$ , we get from (10) that

$$s \int_0^\infty (q(y))^{\alpha-1} \phi_\alpha(q(y)) \exp(-sy) dq(y) \rightarrow \gamma \Gamma(\alpha), \quad \text{as } s \rightarrow \infty.$$

By Karamata's Tauberian theorem, see Theorem 1.7.1' on page 38 of [1], we conclude that

$$v^{-1} \int_0^v (q(x))^{\alpha-1} \phi_\alpha(q(x)) dq(x) \rightarrow \gamma \Gamma(\alpha), \quad \text{as } v \searrow 0,$$

which, in turn, by the change of variable  $y = q(x)$  gives

$$\frac{\int_0^t y^{\alpha-1} \phi_\alpha(y) dy}{-\log \phi_0(t)} \rightarrow \gamma \Gamma(\alpha), \quad \text{as } t \searrow 0.$$

Now using that  $-\log \phi_0(t) \sim 1 - \phi_0(t)$  as  $t \rightarrow 0$ , we end up with

$$\lim_{t \rightarrow 0} \frac{\int_0^t y^{\alpha-1} \phi_\alpha(y) dy}{1 - \phi_0(t)} = \gamma \Gamma(\alpha).$$

Since  $\phi_\alpha(y) = \int_0^\infty e^{-uy} u^\alpha G(du)$ , by Fubini's theorem

$$\begin{aligned}\int_0^t y^{\alpha-1} \phi_\alpha(y) dy &= \int_0^\infty u^\alpha G(du) \int_0^t y^{\alpha-1} e^{-uy} dy \\ &= \int_0^\infty G(du) \int_0^{ut} z^{\alpha-1} e^{-z} dz \\ &= \int_0^\infty \overline{G}(z/t) z^{\alpha-1} e^{-z} dz \\ &= t^\alpha \int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut} du.\end{aligned}$$

A partial integration gives

$$1 - \phi_0(t) = t \int_0^\infty \overline{G}(u) e^{-ut} du.$$

So (10) reads

$$t^{\alpha-1} \frac{\int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut} du}{\int_0^\infty \overline{G}(u) e^{-ut} du} \rightarrow \gamma \Gamma(\alpha), \text{ as } t \searrow 0. \quad (11)$$

From now on we shall assume that (6) holds with  $0 < \gamma \leq 1$ . Let us define the function for  $t > 0$

$$f(t) = \int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut} du. \quad (12)$$

Clearly,  $f$  is monotone decreasing and since  $\mathbb{E}Y = \infty$ ,  $\lim_{t \rightarrow 0} f(t) = \infty$ . Moreover, showing that  $f$  is regularly varying at zero implies that  $\overline{G}$  is regularly varying at infinity. We use the identity

$$u^{1-\alpha} e^{-ut} = \frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2} e^{-(y+t)u} dy,$$

which holds for  $u > 0$  and  $\alpha \in (1, 2]$ . (This is the *Weyl-transform*, or *Weyl-fractional integral* of the function  $e^{-ut}$ .) This identity combined with Fubini's theorem (everything is nonnegative) gives

$$\begin{aligned} \frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2} f(y+t) dy &= \int_0^\infty \overline{G}(u) u^{\alpha-1} du \frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2} e^{-(y+t)u} dy \\ &= \int_0^\infty \overline{G}(u) e^{-ut} du. \end{aligned}$$

So we can rewrite (11) as

$$\lim_{t \searrow 0} \frac{t^{\alpha-1} f(t)}{\int_0^\infty y^{\alpha-2} f(t+y) dy} = \frac{\gamma \Gamma(\alpha)}{\Gamma(\alpha-1)} = \gamma(\alpha-1). \quad (13)$$

A change of variable gives

$$\int_0^\infty y^{\alpha-2} f(t+y) dy = t^{\alpha-1} \int_1^\infty (u-1)^{\alpha-2} f(ut) du,$$

and so we have

$$\lim_{t \searrow 0} \int_1^\infty (u-1)^{\alpha-2} \frac{f(ut)}{f(t)} du = [\gamma(\alpha-1)]^{-1}. \quad (14)$$

We can rewrite  $f$  as

$$f(t) = \int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut} du = t^{-\alpha} \int_0^\infty \overline{G}(u/t) u^{\alpha-1} e^{-u} du,$$

from which we see that the function

$$g(t) = \int_0^\infty \overline{G}(u/t) u^{\alpha-1} e^{-u} du = t^\alpha f(t)$$



is bounded and nondecreasing. Substituting  $g$  into (14) we obtain

$$\lim_{t \rightarrow 0+} \int_1^\infty (u-1)^{\alpha-2} u^{-\alpha} \frac{g(ut)}{g(t)} du = [\gamma(\alpha-1)]^{-1}. \quad (15)$$

Write  $g_\infty(x) = g(x^{-1})$ ,  $x > 0$ . Then (15) has the form

$$\int_1^\infty (u-1)^{\alpha-2} u^{-\alpha} \frac{g_\infty(x/u)}{g_\infty(x)} du = \frac{k^M * g_\infty(x)}{g_\infty(x)} \rightarrow [\gamma(\alpha-1)]^{-1}, \quad \text{as } x \rightarrow \infty, \quad (16)$$

where

$$k(u) = \begin{cases} (u-1)^{\alpha-2} u^{-\alpha+1}, & u > 1, \\ 0, & 0 < u \leq 1, \end{cases}$$

and

$$k^M * h(x) = \int_0^\infty h(x/u) k(u) / u du$$

is the *Mellin-convolution* of  $h$  and  $k$ . Note that the *Mellin-transform* of  $k$ ,

$$\begin{aligned} \tilde{k}(z) &= \int_1^\infty (u-1)^{\alpha-2} u^{-\alpha-z} du = \int_0^1 (1-v)^{\alpha-2} v^z dv \\ &= \frac{\Gamma(\alpha-1) \Gamma(1+z)}{\Gamma(\alpha-z)} = \text{Beta}(\alpha-1, 1+z) \end{aligned}$$

is convergent for  $z > -1$ . We apply a version of the Drasin-Shea theorem (Theorem 5.2.3 on page 273 of [1]). To do this we must verify the following conditions:

1.  $\tilde{k}$  has a maximal convergent strip  $a < \Re z < b$  such that  $a < 0$  and  $b > 0$ ,  $\tilde{k}(a+) = \infty$  and  $\tilde{k}(b-) = \infty$  if  $b < \infty$ . Our  $\tilde{k}$  satisfies this condition with  $a = -1$  and  $b = \infty$ .
2. Our function of interest is

$$g_\infty(x) = g(x^{-1}) = \int_0^\infty \overline{G}(ux) u^{\alpha-1} e^{-u} du, \quad x > 0,$$

is certainly positive and locally bounded.

3. Also our function  $g_\infty$  is of bounded decrease, since for  $\lambda > 1$

$$\frac{g_\infty(\lambda x)}{g_\infty(x)} = \lambda^{-\alpha} \frac{(\lambda x)^\alpha g(1/(\lambda x))}{x^\alpha g(1/x)} = \lambda^{-\alpha} \frac{f(1/(\lambda x))}{f(1/x)} \geq \lambda^{-\alpha},$$

so its lower Matuszewska index is at least  $-\alpha$ .

Therefore by Theorem 5.2.3 of [1], whenever,

$$\frac{k^M * g_\infty(x)}{g_\infty(x)} \rightarrow c, \quad \text{as } x \rightarrow \infty, \quad (17)$$

then  $\tilde{k}(\rho) = c$  for some  $\rho \in (-1, \infty)$ . (In our case by (16),  $c = [\gamma(\alpha - 1)]^{-1}$ .) Moreover, since  $\tilde{k}(z)$  is strictly decreasing on  $(-1, \infty)$  and  $\tilde{k}(0) = \frac{1}{\alpha-1}$ , for any  $0 < \gamma \leq 1$  the solution  $\rho$  to  $\tilde{k}(\rho) = [\gamma(\alpha - 1)]^{-1}$  must lie in  $(-1, 0]$ . Theorem 5.2.3 of [1] also says that  $g_\infty(x)$  is regularly varying at infinity with index  $0 \geq \rho > -1$ .

Next since  $g_\infty(x) = g(x^{-1}) = x^{-\alpha} f(x^{-1}) \in \mathcal{RV}_\infty(\rho)$ , where  $\tilde{k}(\rho) = c$ ,  $g \in \mathcal{RV}_0(-\rho)$ , which implies that  $f \in \mathcal{RV}_0(-\rho - \alpha)$ . Recalling that

$$f(t) = \int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut} du,$$

the Karamata Tauberian theorem now gives that

$$\int_0^x \overline{G}(u) u^{\alpha-1} du \in \mathcal{RV}_\infty(\alpha + \rho).$$

Thus by Lemma 2,  $\overline{G}(u) \in \mathcal{RV}_\infty(\rho)$ .

This says that  $Y \in D(\beta)$ , where  $\rho = -\beta \in (-1, 0]$  and  $\beta$  is the unique solution of

$$\text{Beta}(\alpha - 1, \beta + 1) = \frac{\Gamma(\alpha - 1)\Gamma(1 + \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.$$

We now turn to the proof of the second part of Proposition 2. First consider the case  $\beta = 0$ . Let  $0 \leq D_n^{(n)} \leq \dots \leq D_n^{(1)}$  denote the order statistics of  $Y_1 / (\sum_{i=1}^n Y_i), \dots, Y_n / (\sum_{i=1}^n Y_i)$ . We see that

$$\mathbb{E} \left( D_n^{(1)} \right)^\alpha \leq \mathbb{E} S_n(\alpha) = \sum_{i=1}^n \mathbb{E} \left( D_n^{(i)} \right)^\alpha \leq \mathbb{E} \left( D_n^{(1)} \right)^{\alpha-1} \leq 1.$$

Now  $D_n^{(1)} \rightarrow_P 1$  if and only if  $Y \in D(0)$ . (See Theorem 1 of Haeusler and Mason [5] and their references.) Thus if  $Y \in D(0)$  then (1.6) holds with  $\gamma = 1$ .

Now assume that  $Y \in D(\beta)$ ,  $0 < \beta < 1$ . In this case, there exists a sequence of positive constants  $\{a_n\}_{n \geq 1}$ , such that  $a_n^{-1} \sum_{i=1}^n Y_i \rightarrow_d U$ , where  $U$  is a  $\beta$ -stable random variable, with characteristic function

$$\mathbb{E} e^{itU} = \exp \left\{ \beta \int_0^\infty (e^{itu} - 1) u^{-\beta-1} du \right\}.$$

Moreover,  $Y^\alpha \in D(\beta/\alpha)$ , and it is easy to check that  $a_n^{-\alpha} \sum_{i=1}^n Y_i^\alpha \rightarrow_d V$ , where  $V$  is a  $\beta/\alpha$ -stable random variable, with cf

$$\mathbb{E} e^{itV} = \exp \left\{ \frac{\beta}{\alpha} \int_0^\infty (e^{itu} - 1) u^{-\beta/\alpha-1} du \right\}.$$

Since

$$\lim_{n \rightarrow \infty} n \mathbb{P}\{Y > a_n u, Y^\alpha > a_n^\alpha v\} = \lim_{n \rightarrow \infty} n \overline{G}(a_n(u \vee v^{1/\alpha})) = u^{-\beta} \wedge v^{-\beta/\alpha} =: \Pi((u, \infty) \times (v, \infty)),$$

for  $u, v \geq 0$ ,  $u + v > 0$ , using Corollary 15.16 of Kallenberg [6] one can show that the joint convergence also holds, and the limiting bivariate Lévy measure is  $\Pi$ . That is

$$\left( a_n^{-1} \sum_{i=1}^n Y_i, a_n^{-\alpha} \sum_{i=1}^n Y_i^\alpha \right) \rightarrow_d (U, V),$$

where the limiting bivariate random vector has cf

$$\mathbb{E} e^{i(sU+tV)} = \exp \left\{ \int_{[0,\infty)^2} \left( e^{i(su+tv)} - 1 \right) \Pi(du, dv) \right\} = \exp \left\{ \beta \int_0^\infty \left( e^{i(su+tu^\alpha)} - 1 \right) u^{-\beta-1} du \right\}.$$

Since  $\mathbb{P}\{U > 0\} = \mathbb{P}\{V > 0\} = 1$ , we obtain

$$S_n(\alpha) \rightarrow_d \frac{V}{U^\alpha}.$$

Thus since  $\mathbb{E} S_n(\alpha) \leq 1$  for all  $n \geq 1$

$$\mathbb{E} S_n(\alpha) \rightarrow \mathbb{E} \left( \frac{V}{U^\alpha} \right).$$

Clearly  $\mathbb{P}\{U < \infty\} = 1$ , which implies that  $0 < \mathbb{E} \left( \frac{V}{U^\alpha} \right) \leq 1$ , and thus by the first part of Proposition 2

$$0 < \gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 + \beta)} < 1.$$

□

**Lemma 2** Suppose that for some  $\alpha \geq 1$ ,  $\rho > -1$  and slowly varying function  $L$  at infinity

$$U(x) := \int_0^x \overline{G}(u) u^{\alpha-1} du = L(x) x^{\alpha+\rho}, \quad x > 0,$$

then

$$\overline{G}(u) \sim (\alpha + \rho) L(u) u^\rho, \quad \text{as } u \rightarrow \infty.$$

*Proof* We shall follow closely the proof the lemma on page 446 of Feller [3]. Choose any  $0 < a < b < \infty$ . We see that

$$\begin{aligned} \frac{U(tb) - U(ta)}{U(t)} &= \int_a^b \frac{\overline{G}(ut) (ut)^{\alpha-1} t}{U(t)} du \\ &= \int_a^b \frac{\overline{G}(ut) (ut)^{\alpha-1} t}{L(t) t^{\alpha+\rho}} du = \int_a^b \frac{\overline{G}(ut) u^{\alpha-1} du}{L(t) t^\rho}. \end{aligned}$$

Since  $\overline{G}$  is nonincreasing and  $\overline{G}(ut) / (L(t) t^\rho)$  is necessarily bounded for each  $u > 0$  as  $t \rightarrow \infty$ , just as in Feller one can apply the Helly-Bray theorem to find a positive sequence  $t_k \rightarrow \infty$  such that for a measurable function  $\psi$  on  $[0, \infty)$ ,  $\overline{G}(ut_k) / L(t_k) t_k^\rho \rightarrow \psi(u)$ , for all continuity points  $u$  of  $\psi$ . This implies that for all  $0 < a < b < \infty$

$$\frac{U(t_k b) - U(t_k a)}{U(t_k)} \rightarrow b^{\alpha+\rho} - a^{\alpha+\rho} = \int_a^b \psi(u) u^{\alpha-1} du.$$

This forces  $\psi(u) u^{\alpha-1} = (\alpha + \rho) u^{\alpha+\rho-1}$ , and since  $\psi$  is independent of any particular positive sequence  $t_k \rightarrow \infty$  defining it,

$$\overline{G}(ut) / (L(ut) (ut)^\rho) \rightarrow \alpha + \rho, \text{ as } t \rightarrow \infty.$$

□

**Acknowledgement.** Kevei's research was funded by a postdoctoral fellowship of the Alexander von Humboldt Foundation.

## References

- [1] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation, Encyclopedia of Mathematics and its Applications*, 27, Cambridge University Press, Cambridge, 1987.
- [2] L. Breiman, On some limit theorems similar to the arc-sin law, *Teor. Veroyatnost. i Primenen.* 10, 351–360, 1965.
- [3] W. Feller, *An introduction to probability theory and its applications*. Vol. II. Second edition John Wiley & Sons, Inc., New York-London-Sydney 1971.
- [4] A. Fuchs, A. Joffe and J. Teugels, Expectation of the ratio of the sum of squares to the square of the sum: exact and asymptotic results. *Teor. Veroyatnost. i Primenen.* 46, 297–310, 2001 (Russian); translation in *Theory Probab. Appl.* 46, 243–255, 2002.
- [5] E. Haeusler and D.M. Mason, On the asymptotic behavior of sums of order statistics from a distribution with a slowly varying upper tail. Sums, trimmed sums and extremes, 355–376, *Progr. Probab.*, 23, Birkhäuser Boston, Boston, MA, 1991.
- [6] O. Kallenberg, *Foundations of modern probability. Second edition. Probability and its Applications*. Springer-Verlag, New York, 2002.
- [7] P. Kevei and D.M. Mason, The asymptotic distribution of randomly weighted sums and self-normalized sums. *Electron. J. Probab.* 17, no. 46, 21 pp, 2012.
- [8] D.M. Mason and J. Zinn, When does a randomly weighted self-normalized sum converge in distribution? *Electron. Comm. Probab.* 10, 70–81, 2005.